

# BURGERS EQUATION WITH POISSON RANDOM FORCING

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**ABSTRACT.** We consider the Burgers equation on the real line with forcing given by Poissonian noise with no periodicity assumption. Under a weak concentration condition on the driving random force, we prove existence and uniqueness of a global solution in a certain class. We describe its basin of attraction that can also be viewed as the main ergodic component for the model. We establish existence and uniqueness of global minimizers associated to the variational principle underlying the dynamics. We also prove the diffusive behavior of the global minimizers on the universal cover in the periodic forcing case.

*Keywords:* Burgers equation; random forcing; Poisson point process; random environment; ergodicity; One Force — One Solution Principle; global solution; one-point attractor; variational principle.

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## 1. INTRODUCTION

The Burgers equation is one of the basic nonlinear evolution equations:

$$(1.1) \quad \partial_t u(t, x) + u(t, x) \cdot \partial_x u(t, x) = f(t, x).$$

Here  $t \in \mathbb{R}$  is the time variable,  $x \in \mathbb{R}$  is the space variable. The equation describes the evolution of velocity vector field  $u(\cdot, \cdot)$  of sticky dust particles in the presence of external potential forcing  $f(t, x) = -\partial_x F(t, x)$ .

Burgers introduced this equation as a turbulence model. Although it was soon discovered that the dynamics governed by (1.1) does not describe turbulence adequately, the equation has naturally appeared in various other contexts, from cosmology to traffic modeling. An informative recent survey on Burgers turbulence is [BK07].

One of the remarkable properties of the Burgers equation is that even if the initial data at time  $t_0$  and the forcing are smooth, the solution of the Cauchy problem typically develops discontinuities or shocks, and if one wants to extend the solution beyond the formation of shock waves, one has to work with generalized solutions. Under mild assumptions on the initial data and forcing, only one of the generalized solutions is physical. This solution is called the entropy or viscosity solution and it can be found using a characterization that is often called the Lax–Oleinik variational principle (see, e.g., [BK07] and references therein). Namely, the solution potential (a

function  $U$  such that  $\partial_x U(t, x) = u(t, x)$  for a.e.  $x \in \mathbb{R}$ ) satisfies

$$(1.2) \quad U(t, x) = \inf_{\gamma: \gamma(t)=x} \left\{ U(t_0, \gamma(t_0)) + \int_{t_0}^t L(s, \gamma(s), \dot{\gamma}(s)) ds \right\}.$$

The expression in the curly brackets is called action, and the infimum of action is taken over all absolutely continuous trajectories  $\gamma$  defined on  $[t_0, t]$  and terminating at  $x$  at time  $t$ . The Lagrangian  $L$  is defined by

$$L(t, x, p) = \frac{p^2}{2} - F(t, x).$$

Following the hydrodynamic interpretation of the Burgers equation, one can identify the action minimizers in (1.2) as the particle trajectories. This kind of representation holds true for more general equation of Hamilton–Jacobi type. The specifics of the Burgers equation is that if  $\gamma^*$  is a unique minimizer in (1.2), then  $u(t, x) = \dot{\gamma}^*(t)$ .

When the forcing is a random field, one has to work with optimization problems for paths accumulating action from a random Lagrangian landscape, so questions about Burgers equations with randomness become random media questions.

The ergodic theory of the Burgers equation with random forcing begins with [EKMS00]. The forcing in [EKMS00] is assumed to be white noise type in time and smooth and periodic in space. Due to the periodicity assumption, the evolution effectively takes place on a circle. The compactness of the circle allows for efficient control of the long time behavior of action minimizers, which leads to constructing attracting global solutions and thus to a complete description of the ergodic components for the dynamics, each one consisting of all velocity profiles with given mean velocity.

This work was extended and streamlined in [IK03] and [GIKP05], where the multidimensional version of the Burgers equation with positive or zero viscosity on a torus was considered. In [Bak07] the ergodic theory for the Burgers equation on a segment with random boundary conditions was developed.

In all these papers the compactness of the domain played an important role. In fact, in the case of unbounded domain with no periodicity assumption, currently there is no complete understanding of the ergodic properties of the Burgers equation. Let us summarize what is known.

In [HK03], the Burgers equation in  $\mathbb{R}^d$  with aperiodic white-noise forcing with certain localization properties was considered. A global solution constructed in the paper was shown to have a basin of attraction containing the zero velocity profile, but no interesting properties of the global solution were established, and the description of the domain of attraction of the global solution was incomplete.

In [Sui05], it was noted that in the absence of periodicity assumptions the long time behavior of solutions can depend on the behavior of the initial condition at infinity in an essential way. In particular, it was shown that

outside the main ergodic component (containing the zero velocity profile), there are solutions with significantly different behavior.

In this paper we introduce a new kind of random forcing for the Burgers equation on the real line with no periodicity assumption. The forcing potential we suggest is given by a Poisson point field. In this model, paths accumulate their action traveling through a cloud of random Poissonian points. Although this model preserves many features of the white noise model, it is easier to analyze and visualize. It also has much in common with the well-known Hammersley process (see, e.g., [AD95]) which has been explicitly used for the analysis of hydrodynamic limit resulting in the Burgers equation in [Sep96].

For the new model we are able to construct a global solution via a limiting procedure seeded at zero initial condition, prove a so called One Force — One Solution Principle (1F1S), and describe the main ergodic component of the system, i.e., the basin of attraction of the global solution.

1F1S for the Burgers equation on the circle is tightly connected to the hyperbolicity of the global action minimizer. In particular, for any two Burgers particles, the distance between their backward trajectories (given by the corresponding one-sided action minimizers) converges to zero. A stronger phenomenon occurs in the case of Poissonian forcing: for any two particles, their backward trajectories will meet at one of the Poissonian points in finite time and coincide from that point on in the reverse time. This stronger form of hyperbolicity may naturally be called hyperhyperbolicity.

The rest of the paper is organized as follows: In Section 2 we introduce the new forcing model based on Poissonian points. In Section 3 we discuss the geometry of foliation of the space-time into particle trajectories under point forcing. In Section 4 we formulate our main results. In Section 5 we construct the global solution. In Section 6 we describe its behavior at infinity. In Section 7, we show that this solution is an attractor and describe its basin of attraction. In Section 8 we study global minimizers. An important part of that section is a Central limit theorem discribing the diffusive behavior of global minimizers for periodic Poissonian forcing.

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## 2. POISSONIAN POINT FORCING

The goal of this section is to describe the model rigorously, so let us now be more precise. The model is based on a Poisson point field and we refer to [Kal86] for an introduction to point processes as random integer-valued measures.

We are working on a complete probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . It is convenient to identify  $\Omega$  with the space of locally finite point configurations  $\omega = \{(s_i, x_i), i \in \mathbb{N}\}$  in space-time  $\mathbb{R} \times \mathbb{R}$ . The sigma-algebra  $\mathcal{F}$  is generated

by maps  $N(B)$  assigning to each  $\omega$  the number of points of  $\omega$  in a bounded Borel set  $B \subset \mathbb{R} \times \mathbb{R}$ . The measure  $\mathsf{P}$  is the distribution of a Poisson point field with intensity measure  $\mu(dt \times dx)$ .

Since we want the forcing to be stationary in time, we shall always assume that the intensity is a product measure:

$$\mu(dt \times dx) = dt \times m(dx).$$

Then for disjoint sets  $B_1, \dots, B_n$ , the random variables  $N(B_1), \dots, N(B_n)$  are independent and Poissonian with parameters  $\mu(B_1), \dots, \mu(B_n)$ .

We will denote the integral term in (1.2) as

$$\mathcal{A}^{t_0, t}(\gamma) = \int_{t_0}^t L(s, \gamma(s), \dot{\gamma}(s))ds = \frac{1}{2} \int_{t_0}^t \dot{\gamma}^2(s)ds - \int_{t_0}^t F(s, \gamma(s))ds,$$

and redefine the contribution from the potential by

$$(2.1) \quad \int_{t_0}^t F(s, \gamma(s))ds = N^{t_0, t}(\gamma),$$

where for a path  $\gamma$  and times  $t_0$  and  $t$  satisfying  $t_0 < t$ ,  $N^{t_0, t}(\gamma) = N_{\omega}^{t_0, t}(\gamma)$  is the number of Poissonian points that  $\gamma$  passes through between  $t_0$  and  $t$ . In other words, each Poissonian point visited by the path contributes  $-1$  to the action. An immediate generalization of our model is a compound Poisson point field where each point comes with a random weight which results in random contributions to the action. In fact, all our results can be extended to that case under reasonable assumptions on the random weights. However, for simplicity we concentrate here on the simple Poisson process.

Definition (2.1) results in the following expression for action accumulated by a path  $\gamma$  between times  $t_0$  and  $t > t_0$ :

$$\mathcal{A}^{t_0, t_1}(\gamma) = \mathcal{A}_{\omega}^{t_0, t}(\gamma) = \frac{1}{2} \int_{t_0}^t \dot{\gamma}^2(s)ds - N_{t_0, t}(\gamma).$$

It is well-known (or can be easily derived from the Euler–Lagrange equations) that in the zero forcing field the minimizers (or particle trajectories) are straight lines. We conclude that between visits to Poissonian points action-minimizing paths are straight lines.

Let us introduce more notation. For two times  $t_0$  and  $t_1$  and two sets  $A_0, A_1 \subset \mathbb{R}$  we denote by  $\Gamma_{t_0, A_0}^{t_1, A_1}$  the set of all piecewise linear paths defined between  $t_0$  and  $t_1$  such that switchings from one linear regime to another happen only at Poissonian points. We also denote the set of action minimizers over  $\Gamma_{t_0, A_0}^{t_1, A_1}$  by  $M_{t_0, A_0}^{t_1, A_1} = M_{t_0, A_0}^{t_1, A_1}(\omega)$ . If  $A_0$  or  $A_1$  consists of one point  $x$  we will often use index  $x$  instead of  $\{x\}$  in these notations.

To define the main random dynamical system we must start with the phase space. First we recall that the natural space of solutions for the Burgers equation consists of piecewise continuous functions  $u$  defined on  $\mathbb{R}$ , with right and left limits at every point, with at most countably many discontinuities, each discontinuity being a downward jump, or shock:  $u(x-) > u(x+)$ .

(The shock absorbs incoming particles on both sides.) We shall impose an additional restriction on these functions to be bounded and will not distinguish between two functions that coincide at all their continuity points. We will denote the resulting factor space by  $\mathbb{U}$ , and often we will abuse the notation writing  $u \in \mathbb{U}$  when  $u$  is a representative of an element of  $\mathbb{U}$ .

We will need a measure of proximity in  $\mathbb{U}$ . We denote the set of continuity points of a function  $h \in \mathbb{U}$  by  $C_h$  and for any  $h_1, h_2 \in \mathbb{U}$  write

$$d(h_1, h_2) = \exp \left[ -\sup \{ r > 0 : h_1(x) = h_2(x), |x| < r, x \in C_{h_1} \cap C_{h_2} \cap B_r \} \right],$$

where  $B_r = [-r, r]$ . If there is no neighborhood of the origin where  $h_1$  and  $h_2$  coincide, we set  $d(h_1, h_2) = 1$ . If  $h_1 \equiv h_2$ , we set  $d(h_1, h_2) = 0$ . Thus defined  $d$  is a metric in  $\mathbb{U}$  taking values in  $[0, 1]$ .

Given  $v \in \mathbb{U}$ , we can define a potential  $V$  so that  $V'(x) = v(x)$  for all  $x$ . For any times  $t_0, t_1$  with  $t_0 < t_1$ , we set

$$(2.2) \quad \Phi_{\omega}^{t_0, t_1} v(x) = \dot{\gamma}^*(t_1),$$

where  $\gamma^*$  is the solution of

$$(2.3) \quad V(\gamma(t_0)) + \mathcal{A}_{\omega}^{t_0, t_1}(\gamma) \rightarrow \min, \quad \gamma \in \Gamma_{t_0, \mathbb{R}}^{t_1, x}.$$

Let us assume that

$$(2.4) \quad m(\mathbb{R}) < \infty,$$

and briefly summarize (without proof) several facts about the Burgers equation solution map  $\Phi$  that apply to the current setting.

**Lemma 2.1.** *If  $h \in \mathbb{U}$ , then with probability 1 the following holds:*

- (1) *For any time interval  $[t_0, t_1]$ , in definition (2.2)–(2.3), the minimizer  $\gamma^*$  (and, consequently, its slope  $\dot{\gamma}^*(t_1)$  at the terminal time  $t_1$ ) is defined uniquely for all  $x \in \mathbb{R}$  except at most countably many points. Every point  $x$  where  $\Phi_{\omega}^{t_0, t_1} v(x)$  is uniquely defined is a continuity point of  $\Phi_{\omega}^{t_0, t_1} v$ . At any point where the minimizer is not unique,  $\Phi_{\omega}^{t_0, t_1} v$  makes a downward jump.*
- (2) *The function  $\Phi_{\omega}^{t_0, t_1} v$  is bounded (in particular, combining this with the first part of this Lemma, we obtain that  $\Phi_{\omega}^{t_0, t_1}$  is a map from  $\mathbb{U}$  to itself).*
- (3) *Moreover, for all  $\omega$ , if  $t_0 \leq t_1 \leq t_2$ ,*

$$(2.5) \quad \Phi_{\omega}^{t_1, t_2} \Phi_{\omega}^{t_0, t_1} v = \Phi_{\omega}^{t_0, t_2} v.$$

**Remark 2.1.** Introducing  $\Phi_{\omega}^t = \Phi_{\omega}^{0, t}$  for  $t \geq 0$ , we can rewrite the cocycle property (2.5) as

$$\Phi_{\omega}^{t_1+t_2} v = \Phi_{\theta^{t_1} \omega}^{t_2} \Phi_{\omega}^{t_1} v, \quad t_1, t_2 \geq 0,$$

where  $\theta^t$  denotes the time shift of the Poissonian point field:  $(s_i, x_i) \mapsto (s_i - t, x_i)$ .

Let us denote by  $\mathcal{F}_A$  the sigma-algebra generated by the restriction of the Poissonian point field to  $A \times \mathbb{R}$  for any set  $A$  of times. Clearly, the random operator  $\Phi_\omega^{t_0, t_1}$  depends only on the realization of the Poisson process between times  $t_0$  and  $t_1$ , i.e., it is measurable w.r.t.  $\mathcal{F}_{[t_0, t_1]}$ .

### 3. GEOMETRY OF SOLUTIONS UNDER POISSONIAN FORCING

Throughout this paper we consider the external forcing that is concentrated on a discrete set of Poissonian points (we will often call them *forcing points*.) This is different from traditionally considered smooth forcing fields, so let us understand the effect of this kind of forcing on the solution.

Let us consider a model situation where a smooth beam of Burgers particles encounters a forcing point at the origin at time 0. Let us assume that at time 0, the velocity vector field near 0 is  $u_0(y) = a + by$ , where  $b > 0$ .

It is clear that for every  $(t, x)$  with  $t > 0$  and  $x$  close to the origin there are two minimizer candidates. The minimizer either passes through the origin or it does not. If it does then (assuming there are no other point sources of forcing) it has to be a straight line connecting the origin to  $(t, x)$ , and the accumulated action is  $A_1(t, x) = x^2/(2t) - 1$ , where  $-1$  is the contribution of the forcing point at the origin, and  $x^2/(2t)$  is the action accumulated while moving with constant velocity  $x/t$  between 0 and  $t$ . If the minimizer does not pass through the origin, then it is a straight line connecting some point  $(0, x_0)$  to  $(t, x)$ . On the one hand, the velocity of the particle associated with the minimizer is  $(x - x_0)/t$ . On the other hand, it has to coincide with  $u_0(x_0) = a + bx_0$ . Therefore, we can find  $x_0 = (x - at)/(1 + bt)$ . Taking into account that  $U_0(x_0) = ax_0 + bx_0^2/2$ , we can compute that the total action of that path is  $A_2(t, x) = (bx^2 + 2ax - a^2t)/(2(1 + bt))$ .

To see which of the two cases is realized for  $(t, x)$  we must compute  $A_1(t, x)$  and  $A_2(t, x)$ . If  $A_1(t, x) < A_2(t, x)$ , then the particle arriving to  $x$  at time  $t$  is at the origin at time 0. If  $A_1(t, x) > A_2(t, x)$ , then the particle arriving to  $x$  at time  $t$  is one of the particles that moved with constant velocity and was a part of the incoming beam. If  $A_1(t, x) = A_2(t, x)$ , then both of these paths are minimizers, and at time  $t$  there is a shock at point  $x$ . The relation  $A_1(t, x) = A_2(t, x)$  can be rewritten as

$$(x - at)^2 = 2t(1 + bt).$$

For small values of  $t$ , the set of points satisfying this relation looks like a parabola  $(x - at)^2 = 2t$ , see Figure 3 where an example with  $a = 1$  and  $b = 1/2$  is shown.

We see that when a Poissonian point appears, it emits a continuum of particles each moving with constant velocity, creating two shock fronts moving (at least for a short time) to the left and right.

It is important to notice that in our model case with a forcing point at the origin,  $u(t, x) = x/t$  for all points connected to the origin by a minimizing segment. It means that for each time  $t$ , the velocity is linear in the domain

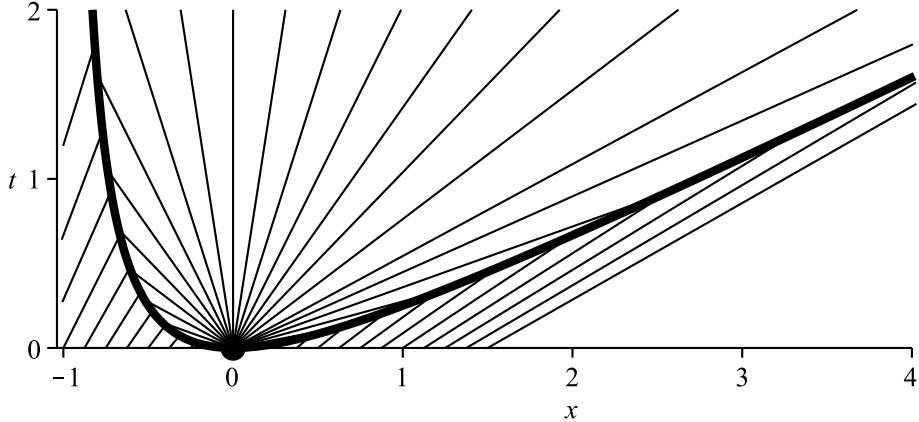


FIGURE 1. Minimizers around a forcing point

of influence of the forcing point, and the velocity gradient decays with time as  $1/t$ .

In general, the behavior of this kind occurs near each forcing point, and in the long run more and more points of the space-time plane get assigned to forcing points. Grouping together points assigned to the same forcing point, we obtain a tessellation of space-time into domains of influence of forcing points. Inside each domain or cell the velocity field is linear in  $x$  if the time  $t$  is fixed.

It is well-known that in the Burgers equation the energy is dissipated at the shocks, see, e.g., [BK07]. By seeding new particles at each Poissonian point, the forcing pumps energy into the system and, therefore, we can hope that there is a dynamical or statistical energy balance in the system. We will actually see that this dissipation results in asymptotic alignment of the velocities of particles that keep moving away from the origin without being absorbed into shocks.

Another point of view at the stationarity and ergodicity issues for this system is related to the stabilization of the tessellation of space-time into cells described above.

#### 4. MAIN RESULT

Although it would be interesting to consider the situation where the spatial intensity measure  $m(dx)$  satisfies  $m(\mathbb{R}) = \infty$ , (e.g., the Lebesgue measure on  $\mathbb{R}$ ), throughout this paper we will adopt either assumption (2.4) or an even stronger finite first moment assumption

$$(4.1) \quad \int_{\mathbb{R}} (1 + |x|)m(dx) < \infty.$$

**Theorem 4.1.** *Suppose (4.1) holds. Then there is a set  $\Omega'$  with  $P(\Omega') = 1$  and a function  $u : \mathbb{R} \times \Omega' \rightarrow \mathbb{U}$  such that on  $\Omega'$  the following holds true:*

- (1)  *$u$  is measurable w.r.t.  $\mathcal{F}_{(-\infty, 0]}$ . In other words, it depends only on  $\omega|_{(-\infty, 0]}$ .*
- (2)  *$u$  defines a global solution (in other words, it is skew-invariant under  $(\Phi, \theta)$ )*

$$\Phi_\omega^t u_\omega = u_{\theta^t \omega}, \quad t \geq 0.$$

- (3) *The solution  $u_\omega$  is piecewise linear.*
- (4) *There is a nonrandom constant  $q > 0$  such that*

$$\lim_{x \rightarrow \pm\infty} u_\omega(x) = \pm q.$$

- (5) *This solution  $u$  plays the role of a one-point attractor. Namely, if  $V' = v \in \mathbb{U}$  and*

$$(4.2) \quad \liminf_{x \rightarrow \infty} \frac{V(x)}{x} > -q,$$

*we have forward attraction:*

$$(4.3) \quad d(\Phi_\omega^t v, u_{\theta^t \omega}) \rightarrow 0, \quad t \rightarrow \infty,$$

*and pullback attraction:*

$$(4.4) \quad d(\Phi_{\theta^{-t} \omega}^t v, u_\omega) \rightarrow 0, \quad t \rightarrow \infty.$$

*The function  $u$  is a unique (up to zero measure modifications) global solution satisfying*

$$(4.5) \quad \liminf_{x \rightarrow \infty} \frac{U_\omega(x)}{x} > -q,$$

*with positive probability (here  $U_\omega$  is the potential of  $u_\omega$ , i.e.,  $U'_\omega \equiv u_\omega$ .)*

**Remark 4.1.** One can reformulate the theorem in terms of a global solution defined as a function of three variables:  $u_\omega(t, x) = u_{\theta^t \omega}(x)$ .

**Remark 4.2.** If one accepts a weaker condition 2.4, then all conclusions of Theorem 4.1 except conclusion 4 still hold and their proofs do not change. Conclusion 4 has to be replaced with a weaker one

$$\lim_{x \rightarrow \infty} \frac{U_\omega(x)}{x} = q.$$

**Remark 4.3.** Conclusion 4 means that in the stationary regime, at infinity one observes particles moving away from the origin with velocity  $q$ .

**Remark 4.4.** Conclusion 5 means that if one starts with an initial condition that sends particles from infinity towards zero with speed that is less than  $q$ , see condition (4.2), then this inbound flow is not strong enough to compete with the outbound flow of particles developed due to the noise, and in the long run it is dominated by the latter. If the condition (4.2) is violated, then the long term properties of solutions are sensitive to the details of the behavior of the initial condition at infinity because the inbound flow

of particles may be stronger than the outbound one, and one will observe effects similar to those discussed in [Sui05].

**Remark 4.5.** The uniqueness conclusion 5 and measurability property (conclusion 1) can be combined into 1F1S Principle — at time 0 there is a unique velocity profile compatible with the history of the forcing.

## 5. CONSTRUCTING A GLOBAL SOLUTION.

In this section we construct a global solution  $u$ . To do this, we start with the zero initial condition at time  $-T$  and take  $T$  to infinity. Our goal is to show that  $\Phi_\omega^{-T,0}$  converges in  $(\mathbb{U}, d)$  to a limiting function and that this limit defines a global solution.

It will be convenient to assume that 0 belongs to the support of measure  $m$ , i.e., for any  $\delta > 0$ ,  $m(B_\delta) > 0$ . We adopt this nonrestrictive assumption without loss of generality since one can always introduce a shift coordinate change to make it hold true.

Since all admissible paths are composed of straight line segments, we will often use the following elementary result on action accumulated along one segment:

**Lemma 5.1.** *A path corresponding to a particle moving with constant velocity  $v$  for time  $t$  and visiting no Poissonian points, accumulates action equal to*

$$\frac{v^2 t}{2} = \frac{vx}{2} = \frac{x^2}{2t},$$

where  $x = vt$  is the traveled distance.

**Lemma 5.2.** *There are numbers  $a, b > 0$  and an a.s.-finite random variable  $\beta > 0$  such that if  $t > 0$ ,  $x \in \mathbb{R}$ , and  $\omega \in \Omega$  satisfy  $t - |x| - 2b \geq \beta(\omega)$ , then there is a path  $\bar{\gamma}$  with  $\bar{\gamma}(-t) = x$  and*

$$\mathcal{A}_\omega^{-t,0}(\bar{\gamma}) < -(t - |x|)a + |x| + b.$$

PROOF: Let us and consider sets  $A_k = [-2k, -2k+1] \times B_{1/2}$ . For any  $k \in \mathbb{N}$  we have

$$\mathbb{P}\{N(A_k) \neq 0\} = 1 - e^{-M},$$

where  $N(A_k)$  denotes the number of Poisson points in  $A_k$  and  $M = m(B_{1/2})$ . For any  $s > 0$  we denote by  $X(s)$  the random number of indices  $k \in \mathbb{N}$  satisfying  $k < s/2$  and  $N(A_k) \neq 0$ . The sequence  $\mathbf{1}_{\{N(A_k) \neq 0\}}$  is i.i.d. with mean  $1 - e^{-M}$ , and the strong law of large numbers implies that there is a random time  $\beta$  such that if  $s > \beta$ , then

$$(5.1) \quad X(s) > s(1 - e^{-M})/3.$$

Consider a path  $\bar{\gamma}$  that starts at  $(-t, x)$  and visits exactly one point in set  $A_k$  if  $k$  satisfies  $N(A_k) \neq 0$  and  $2k < t - |x| - 1$ , and no other points. Each Poissonian point in the path contributes  $-1$  to the action, and we can use Lemma 5.1 to see that each segment connecting these points contributes at

most  $(2 \cdot 1/2)^2/2 = 1/2$ . The slope of the segment with endpoint  $(-t, x)$  does not exceed 1, and contributes at most  $(|x| + 1)/2$  to the action.

If  $t - |x| - 1 > \beta$  then we can combine this with (5.1) applied to  $t - |x| - 1$  and obtain

$$\begin{aligned} \mathcal{A}_\omega^{-t,0}(\bar{\gamma}) &< -(t - |x| - 1) \frac{1 - e^{-M}}{3} \frac{1}{2} + (|x| + 1)/2 \\ &< -(t - |x|)(1 - e^{-M})/6 + |x| + (1 - e^{-M})/6 + 1/2, \end{aligned}$$

and the lemma follows with  $a = (1 - e^{-M})/6$  and  $b = (1 - e^{-M})/6 + 1/2 > 1/2$  since  $t - |x| - 2b > \beta$  implies  $t - |x| - 1 > \beta$ .  $\square$

Let us recall that  $B_r = [-r, r]$  for any  $r > 0$ . The following is the main localization lemma.

**Lemma 5.3.** *There are random variables  $r^-$ ,  $r^+$ ,  $r^\pm$ ,  $(\tau_R^-)_{R>0}$ ,  $(\tau_R^+)_{R>0}$ ,  $(\tau_R^\pm)_{R>0}$ , such that for any  $R > 0$ ,*

$$(5.2) \quad \mathbb{P} \left\{ \text{there are } t > \tau_R^-, x \in B_R, \text{ and } \gamma \in M_{-t,x}^{0,\mathbb{R}} \text{ s.t. } |\gamma(0)| > r^- \right\} = 0;$$

$$(5.3) \quad \mathbb{P} \left\{ \text{there are } t > \tau_R^+, x \in B_R, \text{ and } \gamma \in M_{0,\mathbb{R}}^{t,x} \text{ s.t. } |\gamma(0)| > r^+ \right\} = 0;$$

$$(5.4) \quad \mathbb{P} \left\{ \text{there are } t_-, t_+ > \tau_R^\pm, x_-, x_+ \in B_R, \text{ and } \gamma \in M_{-t_-,x_-}^{t_+,x_+} \text{ s.t. } |\gamma(0)| > r^\pm \right\} = 0.$$

Additionally, there are random variables  $(\bar{\tau}_R)_{R>0}$  and a number  $R' > 0$  such that for any  $R > R'$ ,

$$(5.5) \quad \mathbb{P} \left\{ \text{there are } t_- > \bar{\tau}_R, t_+ > \tau_R^+, x_+ \in B_R, \text{ and } \gamma \in M_{-t_-,x_-}^{t_+,x_+} \text{ s.t. } |\gamma(0)| > r^\pm \right\} = 0.$$

**Remark 5.1.** The idea of this lemma is that minimizers over long time intervals are localized within a random neighborhood of the origin. Each of the random variables  $r^-$ ,  $r^+$ , and  $r^\pm$  can be called localization radius.

PROOF: Let us prove (5.2) first. We are going to construct random variables  $K$  and  $h$  so that for any  $R > 0$  and for sufficiently large  $t$ , no path  $\gamma$  with  $|\gamma(0)| > Kh$  can belong to  $M_{-t,x}^{0,\mathbb{R}}$  with  $x \in B_R$ . The reason why we need two random variables is that we will use  $h$  as an intermediate threshold.

Let  $x \in B_R$ . Consider a path  $\gamma$  defined on  $[-t, 0]$  such that  $|\gamma(0)| > Kh$  and  $\gamma(-t) = x$ . Suppose that  $|\gamma(-s)| \leq h$  for some  $s \in [0, t]$  and define  $\sigma = \sup\{s \leq t : |\gamma(-s)| \geq h\}$ . Then

$$\mathcal{A}_\omega^{-\sigma,0}(\gamma) \geq \frac{(K-1)^2 h^2}{2\sigma} - N([-s, 0] \times B_h^c).$$

To treat the second term in the r.h.s., we need the following result:

**Lemma 5.4.** *For any  $\varepsilon > 0$ , there is a positive random variable  $R_0$  such that with probability 1, for every  $t > 0$*

$$N([-t, 0] \times B_{R_0}^c) < \varepsilon t.$$

PROOF: Let us choose a number  $\alpha_1$  such that  $m(B_{\alpha_1}^c) < \varepsilon/2$ . Due to the strong law of large numbers, there is a random time  $\tau > 0$  such that  $N([-t, 0] \times B_{\alpha_1}^c) < \varepsilon t$  for all  $t > \tau$ . With probability 1, there are finitely many Poissonian points in  $[-\tau, 0] \times \mathbb{R}$ . Let  $\alpha_2$  be the maximal absolute value of the spatial components of these points. The conclusion of the lemma holds true with  $R_0 = \alpha_1 \vee \alpha_2$ .  $\square$

Coming back to the proof of (5.2), let us set  $\varepsilon = a/2$ , where  $a$  is defined in Lemma 5.2. Lemma 5.4 applied to this value of  $\varepsilon$ , ensures the existence of  $h = h(\omega)$  such that

$$(5.6) \quad \mathcal{A}_\omega^{-\sigma,0}(\gamma) \geq \frac{(K-1)^2 h^2}{2\sigma} - \varepsilon\sigma.$$

If  $\sigma < (K-1)h/\sqrt{2\varepsilon}$ , then  $\mathcal{A}_\omega^{-\sigma,0}(\gamma) > 0$ , and the comparison with a zero velocity trajectory with zero action proves that  $\gamma$  cannot be a minimizer.

To treat the case where

$$(5.7) \quad \sigma \geq (K-1)h/\sqrt{2\varepsilon},$$

we will impose some restrictions on  $K$ . First, we require that  $K(\omega) \geq K_1(\omega)$ , where

$$K_1(\omega) = \left( \frac{\beta(\omega) + 2b}{h(\omega)} + 1 \right) \sqrt{2\varepsilon} + 2,$$

with  $\beta$  and  $b$  constructed in Lemma 5.2.

Then, under assumption (5.7),  $\sigma - h - 2b > \beta$  and we can apply Lemma 5.2. The path  $\bar{\gamma}$  constructed in that lemma for point  $(-\sigma, \gamma(-\sigma))$  satisfies

$$\mathcal{A}_\omega^{-\sigma,0}(\bar{\gamma}) \leq -(\sigma - h)a + h + b.$$

On the other hand, (5.6) implies

$$\mathcal{A}_\omega^{-\sigma,0}(\gamma) \geq -\varepsilon\sigma \geq -a\sigma/2.$$

The last two inequalities imply

$$\begin{aligned} \mathcal{A}_\omega^{-\sigma,0}(\bar{\gamma}) - \mathcal{A}_\omega^{-\sigma,0}(\gamma) &\leq -(\sigma - h)a + h + b + a\sigma/2 \\ &\leq -a\sigma/2 + h(a + 1) + b, \end{aligned}$$

and, due to (5.7), the r.h.s. is negative if we assume that  $K(\omega) \geq K_2(\omega)$ , where

$$K_2(\omega) = \frac{2\sqrt{2\varepsilon}(h(\omega)(a + 1) + b)}{ah(\omega)} + 2.$$

Therefore, under this assumption  $\gamma$  cannot be a minimizer. We conclude that if  $K > K_1 \vee K_2$ , then  $\gamma$  with  $|\gamma(0)| > Kh$  cannot be a minimizer satisfying  $|\gamma(-s)| \leq h$  for some  $s \in [0, t]$ .

Let us now consider a path  $\gamma$  with  $|\gamma(-s)| > h$  for all  $s \in [0, t]$ . We have then

$$\mathcal{A}_\omega^{-t,0}(\gamma) \geq -\varepsilon t \geq -at/2.$$

On the other hand we can invoke Lemma 5.2 to see that if  $t - |x| - 2b > \beta$  then there is a path  $\bar{\gamma}$  with  $\bar{\gamma}(-t) = x$  such that

$$\mathcal{A}_\omega^{-t,0}(\bar{\gamma}) < -(t - |x|)a + |x| + b,$$

and

$$\begin{aligned} \mathcal{A}_\omega^{-t,0}(\bar{\gamma}) - \mathcal{A}_\omega^{-t,0}(\gamma) &\leq -(t - |x|)a + |x| + b + at/2 \\ &\leq -at/2 + (a + 1)|x| + b. \end{aligned}$$

Since  $|x| \leq R$ , the r.h.s. is negative if we require that

$$t > \frac{2(R(a + 1) + b)}{a}.$$

Under this additional assumption,  $\gamma$  cannot be a minimizer. We conclude that (5.2) holds if one chooses

$$r^-(\omega) = (K_1(\omega) \vee K_2(\omega))h(\omega)$$

and

$$\tau_R^-(\omega) = \frac{2(R(a + 1) + b)}{a} \vee (\beta(\omega) + R + 2b).$$

The second part of the lemma, equation (5.3), is only a time reversed version of the first one. The proof of (5.4) is an adaptation of the above argument to the two-sided situation.

Let us prove (5.5). First, choose  $R'$  large enough to ensure that due to the law of large numbers, an optimal path cannot stay infinitely outside  $B_{R'}$ . Therefore, for sufficiently large  $t_-$ , minimizers from  $M_{-t_-}^{t_+, x_+}$  visit a point  $x_- \in B_{R'} \subset B_R$  between  $-t_-$  and  $-\tau_R$ . Since  $x_-, x_+ \in B_R$  and a restriction of a minimizer is a minimizer itself, we can finish the proof by invoking (5.4).  $\square$

Let us denote

$$\begin{aligned} r(\omega) &= r^-(\omega) \vee r^+(\omega) \vee r^\pm(\omega), \\ \tau_R(\omega) &= \tau_R^-(\omega) \vee \tau_R^+(\omega) \vee \tau_R^\pm(\omega), \quad R > 0, \\ D_1(R, T) &= \{r(\omega) < R, r(\theta^T \omega) < R, \tau_R(\omega) < T, \tau_R(\theta^T \omega) < T\}, \quad R, T > 0. \end{aligned}$$

**Lemma 5.5.** *For any  $L > 0$  there are numbers  $R > L$  and  $T > 0$  such that  $P(D_1(R, T)) > 0$ .*

PROOF: We take  $R$  so large that  $P\{r(\omega) > R\} < 1/4$ . Then  $P\{r(\theta^t \omega) > R\} < 1/4$  for any  $t$  since  $\theta^t$  preserves the measure. Then we take  $T$  so large that  $P\{\tau_R(\omega) > T\} < 1/4$ . Then  $P\{\tau_R(\theta^T \omega) > T\} < 1/4$ , and the lemma follows.  $\square$

Let us fix the values of  $R$  and  $T$  given by Lemma 5.5 and introduce a new event  $D_2(R, T)$  consisting of all outcomes  $\omega$  admitting a point  $(t^*, x^*) = (t^*, x^*)(\omega) \in [0, T] \times \mathbb{R}$  such that for any  $x, y \in B_R$ , the optimal path connecting  $(0, x)$  and  $(T, y)$  passes through  $(t^*, x^*)$ .

**Lemma 5.6.** *Let  $R$  and  $T$  be provided by Lemma 5.5. Then*

$$\mathbb{P}(D_1(R, T) \cap D_2(R, T)) > 0.$$

PROOF: The proof of this lemma is based on a resampling of the point configurations in  $[0, T] \times \mathbb{R}$  according to a certain kernel. In this proof it is convenient to represent  $\omega \in \Omega$  as  $\omega = (\omega_{in}, \omega_{out})$  where  $\omega_{in} \in \Omega_{in}$  and  $\omega_{out} \in \Omega_{out}$  are restrictions of the point configuration  $\omega$  to  $[0, T] \times \mathbb{R}$  and its complement. We also denote by  $\mathbb{P}_{in}$  and  $\mathbb{P}_{out}$  the distributions of Poisson point field in  $[0, T] \times \mathbb{R}$  and its complement in  $\mathbb{R} \times \mathbb{R}$ .

We will take a large number  $n$  and consider a family of rectangles  $L_k, k = 1, \dots, n$  in  $[0, T] \times \mathbb{R}$ . We postpone a precise description of these rectangles.

For every  $\omega \in D_1 = D_1(R, T)$  we consider a new random configuration  $\omega'$ . It coincides with  $\omega$  outside of  $[0, T] \times \mathbb{R}$  and the restriction of  $\omega'$  onto  $[0, T] \times \mathbb{R}$  consists of  $n$  independent random points such that for each  $k = 1, \dots, n$ , the distribution of  $k$ -th point is concentrated in  $L_k$ ,  $k = 1, \dots, n$ . Let us denote the distribution of the configuration of these  $n$  points in  $[0, T] \times \mathbb{R}$  by  $\mathbb{P}'_{in}$ . Later, we shall choose the distributions of individual points appropriately to make  $\mathbb{P}'_{in}$  absolutely continuous w.r.t.  $\mathbb{P}_{in}$ .

To define the resampling more formally, for any  $\omega$  we consider a version of conditional probability  $\mathbb{P}(\cdot | \omega_{out})$  defined for a set  $D$  by

$$\mathbb{P}(D | \omega_{out}) = \mathbb{P}_{in}\{\omega'_{in} : (\omega'_{in}, \omega_{out}) \in D\},$$

and define a new measure  $\mathbb{P}'$  via

$$\mathbb{P}'(E | \omega_{out}) = \mathbb{P}(D_1 | \omega_{out}) \mathbb{P}'_{in}\{\omega'_{in} : (\omega'_{in}, \omega_{out}) \in E\}$$

and

$$\mathbb{P}'(E) = \int_{\Omega_{out}} \mathbb{P}_{out}(d\omega_{out}) \mathbb{P}'(E | \omega_{out}).$$

Let us prove that  $\mathbb{P}' \ll \mathbb{P}$ . We must show that for any set  $E$  with  $\mathbb{P}'(E) > 0$ , we have  $\mathbb{P}(E) > 0$ . Since  $\mathbb{P}'(E) > 0$ , the definition of  $\mathbb{P}'$  yields

$$(5.8) \quad \mathbb{P}_{out}\{\omega_{out} : \mathbb{P}'(E | \omega_{out}) > 0\} > 0.$$

Notice that if  $\omega_{out}$  satisfies  $\mathbb{P}'(E | \omega_{out}) > 0$  then  $\mathbb{P}(D_1 | \omega_{out}) > 0$  and  $\mathbb{P}'_{in}\{\omega'_{in} : (\omega'_{in}, \omega_{out}) \in E\} > 0$ . The latter and the absolute continuity of  $\mathbb{P}'_{in}$  w.r.t.  $\mathbb{P}_{in}$  imply that  $\mathbb{P}_{in}\{\omega'_{in} : (\omega'_{in}, \omega_{out}) \in E\} > 0$  for such  $\omega_{out}$ . Therefore, due to (5.8),

$$\mathbb{P}(E) = \mathbb{P}_{out} \times \mathbb{P}_{in}(E) = \int_{\Omega} \mathbb{P}_{out}(d\omega_{out}) \mathbb{P}_{in}\{\omega'_{in} : (\omega'_{in}, \omega_{out}) \in E\} > 0,$$

and the absolute continuity is proven.

Therefore,  $\mathbb{P}(D_1(R, T) \cap D_2(R, T)) > 0$  will hold if

$$(5.9) \quad \mathbb{P}'(D_1(R, T) \cap D_2(R, T)) > 0.$$

So, it remains to finish the construction of the measure  $\mathbb{P}'_{in}$  and ensure that (5.9) holds along with  $\mathbb{P}'_{in} \ll \mathbb{P}_{in}$ .

We know that  $\mathsf{P}(D_1(R, T)) > 0$  and therefore there are numbers  $l \in \mathbb{N}$  and  $\Delta, M > 0$  such that

$$(5.10) \quad \mathsf{P}(D_1(R, T, l, M, \Delta)) > 0,$$

where

$$D_1(R, T, l, M, \Delta) = D_1(R, T) \cap \{N([0, T] \times \mathbb{R}) = N([\Delta, T - \Delta] \times B_M) = l\}.$$

We let  $n \in \mathbb{N}$  be a large number and  $\delta \in (0, 1/2)$  a small number to be chosen later and define  $L_k = J_k \times B_\delta$ ,  $k = 1, \dots, n$ , where  $J_k = [(2k-1)T/(2n+1), 2kT/(2n+1)]$ . The measure  $\mathsf{P}'_{in}$  on configurations in  $[0, T] \times \mathbb{R}$  is defined as follows: all configurations consist of exactly  $n$  independent points,  $k$ -th point distributed independently in  $L_k$  according to  $(2n+1)/(m(B_\delta)T)m(dx)dt$ . Equivalently, we can say that  $\mathsf{P}'_{in}$  is the distribution of the original Poissonian point field conditioned on having exactly one point in each  $L_k$ . Thus, the absolute continuity property  $\mathsf{P}'_{in} \ll \mathsf{P}_{in}$  holds and it remains to prove (5.9). Taking into account (5.10), it is sufficient to show that for any  $\omega \in D_1(R, T, l, M, \Delta)$ ,

$$(5.11) \quad \mathsf{P}'_{in} \{ \omega'_{in} : (\omega'_{in}, \omega_{out}) \in D_1(R, T) \cap D_2(R, T) \} = 1.$$

First, let us prove that resampled point configurations belong to  $D_1(R, T)$ . Since resampling happens only inside  $[0, T] \times \mathbb{R}$ , the time  $\tau_R^-$  (depending only on the realization in  $(-\infty, 0] \times \mathbb{R}$ ) does not change. Therefore  $r^-(\omega') < R$  and  $\tau_R^-(\omega') < T$ , where  $\omega' = (\omega'_{in}, \omega_{out})$ .

Let us prove that  $r^+(\omega') < R$  and  $\tau_R^+(\omega') < T$ . We need to show that for any  $y \in B_R$  and any  $t > T$ , any  $\gamma' \in M_{0, \mathbb{R}}^{t, y}(\omega')$  satisfies  $\gamma'(0) \in B_R$ . This is certainly true if  $\gamma'$  passes through a point of  $\omega'_{in}$ . So, let us assume that it does not pass through any points of  $\omega'_{in}$ . Therefore, between 0 and  $T$  it is a straight line. Consider now a path  $\gamma \in M_{0, \mathbb{R}}^{t, y}(\omega)$ . We know that  $\gamma(0) \in B_R$ . Let  $t_0 = \sup\{t \in [0, T] : \gamma(t) \in B_R\}$ . Due to the definition of  $D_1(R, T, l, M, \Delta)$ ,  $t_0 > \Delta$ .

Let the path  $\bar{\gamma}$  visit all available points in  $\omega'_{in}$  between 0 and  $t_0/2$ , then move straight to  $(t_0, \gamma'(t_0))$  and coincide with  $\gamma$  after  $t_0$ . We are going to show that  $\mathcal{A}_{\omega'}^{0, t}(\bar{\gamma}) < \mathcal{A}_{\omega'}^{0, t}(\gamma')$  so that  $\gamma'$  cannot be a minimizer. Since  $\gamma'$  does not pass through any points of  $\omega'_{in}$ ,

$$(5.12) \quad \mathcal{A}_{\omega'}^{0, t}(\gamma') \geq \mathcal{A}_{\omega}^{0, t}(\gamma') \geq \mathcal{A}_{\omega}^{0, t}(\gamma) \geq \mathcal{A}_{\omega'}^{0, t}(\bar{\gamma}) + (\mathcal{A}_{\omega}^{0, t}(\gamma) - \mathcal{A}_{\omega'}^{0, t}(\bar{\gamma})).$$

Let us estimate the difference in the r.h.s. Switching from  $\gamma$  to  $\bar{\gamma}$  we lose at most  $l$  Poissonian points, but what do we gain? The action of a path visiting  $r$  points from  $\omega'_{in}$  in a row does not exceed

$$A(r) = r \frac{(2\delta)^2}{2T/(2n+1)} - r,$$

and, since there are at least  $nt_0/(3T)$  points visited by  $\bar{\gamma}$  in  $\omega'_{in}$  between 0 and  $t_0$ ,

$$\mathcal{A}_{\omega'}^{0,t_0}(\bar{\gamma}) < \frac{nt_0}{3T} \left( \frac{(2\delta)^2}{2T/(2n+1)} - 1 \right) + \frac{R^2}{2(t_0/2)}.$$

Therefore,

$$\mathcal{A}_{\omega'}^{0,t}(\gamma) - \mathcal{A}_{\omega'}^{0,t}(\bar{\gamma}) \geq -l + \frac{nt_0}{3T} \left( 1 - \frac{(2\delta)^2}{2T/(2n+1)} \right) - \frac{R^2}{2(t_0/2)}.$$

Choosing  $n$  to be large and  $\delta$  small we see that the r.h.s. is positive, which in conjunction with (5.12) gives the desired inequality  $\mathcal{A}_{\omega'}^{0,t}(\bar{\gamma}) < \mathcal{A}_{\omega'}^{0,t}(\gamma')$ .

This finishes the proof of  $r^+(\omega') < R$  and  $\tau_R^+(\omega') < T$ . It is also easy to adjust the above argument to show that  $r(\omega') < R$  and  $\tau_R(\omega') < T$ , and in the same way one can prove that  $r(\theta^T \omega') < R$  and  $\tau_R(\theta^T \omega') < T$ . Thus,  $\omega' \in D_1(R, T)$ , and it remains to prove that  $\omega' \in D_2(R, T)$  a.s.

Let us prove the following claim: for any points  $x, y \in B_R$ , an optimal path  $\gamma \in M_{x,0}^{y,T}(\omega')$  cannot avoid all points of  $\omega'$  between 0 and  $T/3$ . In fact if it does avoid these points, then

$$(5.13) \quad \mathcal{A}_{\omega'}^{0,T}(\gamma) \geq -\frac{2}{3}n - 1.$$

On the other hand, consider the path  $\bar{\gamma} \in \Gamma_{0,x}^{T,y}$  that visits all points of  $\omega'$  between  $T/8$  and  $7T/8$ .

$$(5.14) \quad \mathcal{A}_{\omega'}^{0,T}(\bar{\gamma}) \leq 2 \frac{(2R)^2}{2T/8} - \left( n \left( \frac{7T}{8} - \frac{T}{8} \right) - 2 \right) \left( 1 - \frac{(2\delta)^2}{2T/(2n+1)} \right),$$

and, for sufficiently small  $\delta$  and large  $n$ ,  $\mathcal{A}_{\omega'}^{0,T}(\bar{\gamma}) < \mathcal{A}_{\omega'}^{0,T}(\gamma)$ , which contradicts our assumption  $\gamma \in M_{x,0}^{y,T}(\omega')$ . Our claim is proven, and in the same way one can prove that an optimal path  $\gamma \in M_{x,0}^{y,T}(\omega')$  must pass through one of the points of  $\omega'$  between  $2T/3$  and  $T$ . Clearly, for sufficiently small  $\delta > 0$  any optimal path passing through a point in  $\omega'$  between 0 and  $T/3$  and a point in  $\omega'$  between  $2T/3$  and  $T$  also passes through all points in  $\omega'$  in between. Therefore,  $\omega' \in D_2(R, T)$ , and the proof of Lemma 5.6 is complete.  $\square$

We can now construct a global solution. For a set  $A$  of paths and a time interval  $[s, t]$  we denote by  $A|_{[s,t]}$  the set of restrictions of all trajectories from  $A$  to  $[s, t]$ .

**Lemma 5.7.** *Let  $R > 0$ . There are two random times  $\sigma_0, \sigma_1 > 0$  such that for all  $x \in B_R$  and any two times  $t_1, t_2 > \sigma_1$*

$$M_{-t_1, \mathbb{R}}^{0,x}(\omega)|_{[-\sigma_0, 0]} = M_{-t_2, \mathbb{R}}^{0,x}(\omega)|_{[-\sigma_0, 0]}.$$

**PROOF:** The ergodicity of  $\theta^1$ , Lemma 5.6 and Poincaré's Recurrence Theorem imply that with probability 1 there is an integer time  $n > T$  such that  $\theta^{-n}\omega \in D_1(R, T) \cap D_2(R, T)$ . Without loss of generality we can assume that

$R > R'$ , where  $R'$  is defined in Lemma 5.3. The last part of that lemma implies that if we define

$$(5.15) \quad \sigma_1(\omega) = n + \bar{\tau}_R(\theta^{-n}\omega) + \bar{\tau}_R(\theta^{-n+T}\omega),$$

then for any  $t > \sigma_1$  and any  $x \in B_R$ , any path  $\gamma \in M_{-t, \mathbb{R}}^{0,x}(\omega)$  satisfies  $\gamma(-n) \in B_{r^\pm(\theta^{-n}\omega)} \subset B_R$  and  $\gamma(-n+T) \in B_{r^\pm(\theta^{-n+T}\omega)} \subset B_R$ . Here we used the fact that  $\theta^{-n}\omega \in D_1(R, T)$ .

Since  $\theta^{-n}\omega \in D_2(R, T)$ , for any  $t_1$  and  $t_2$  satisfying (5.15) and any  $x \in B_R$ , any paths  $\gamma_1 \in M_{-t_1, \mathbb{R}}^{0,x}(\omega)$  and  $\gamma_2 \in M_{-t_2, \mathbb{R}}^{0,x}(\omega)$  pass through a common point at some time  $\sigma_0 \in [-n, -n+T]$ . Therefore, the restrictions of the sets of minimizers on  $[-\sigma_0, 0]$  coincide, and the proof is complete.  $\square$

**Remark 5.2.** In fact, there is an infinite strictly increasing sequence  $(n_k)_{k \in \mathbb{N}}$  such that  $\theta^{-n_k}\omega \in D_1(R, T) \cap D_2(R, T)$  for all  $k$ . Therefore, the theorem can be strengthened. Its conclusion holds for any (random or deterministic)  $\sigma_0 > 0$ . In particular, the finite time minimizers stabilize to a limiting infinite one-sided minimizer.

We can now finish our construction of the global solution  $u$ . For any  $r \in \mathbb{R}$  and  $R > 0$ , the restriction of  $\Phi^{-s, t}0$  on  $B_R$  stabilizes for large values of  $s$ , and

$$u_\omega(t) = (\mathbb{U}, d) \lim_{s \rightarrow \infty} \Phi^{-s, t}0$$

is well defined. Clearly, the construction of  $u_\omega(t)$  depends only on the restriction of  $\omega$  on  $(-\infty, t] \times \mathbb{R}$ .

Since restrictions of minimizers are also minimizers, we can deduce that for any time interval  $[t_0, t_1]$ ,  $u_\omega(t_1)$  is the solution of the Cauchy problem with initial value  $u_\omega(t_0)$ . Therefore, thus constructed function  $u_\omega$  is a global solution of the Burgers equation corresponding to the realization of the random forcing  $\omega$ . This proves parts 1 and 2 of the main theorem.

Let us prove part 3 of the main theorem. For each point  $x$  of continuity of  $u_\omega$ , we denote  $\pi_\omega(x)$  the Poissonian point that is visited last by the minimizer  $\gamma \in M_{-t, \mathbb{R}}^{0,x}$  for sufficiently large  $t$ . The map  $\pi_\omega$  is piecewise constant. If  $\pi_\omega(x) = (s_i, x_i)$  for all  $x$  in an interval  $J$ , then

$$u_\omega(x) = \frac{x - x_i}{|s_i|}, \quad x \in J,$$

and part 3 follows.

## 6. THE BEHAVIOR OF GLOBAL SOLUTION $u(t, x)$ AS $x \rightarrow \infty$ .

In this section we prove part 4 of the main theorem. We will concentrate on proving the limit behavior as  $x \rightarrow +\infty$  since the limit  $x \rightarrow -\infty$  can be studied in exactly the same way.

The idea is that if we want to consider, say, a path in  $M_{0, \mathbb{R}}^{T,x}$  for large values of  $x$  and  $T$ , then the path naturally decomposes into two parts. Most Poissonian points are scattered over a compact domain, so in a certain time

interval  $[0, t]$  the path mostly stays in a compact domain around the origin collecting action at approximately linear rate  $S < 0$ , and then it leaps from the compact domain straight to  $x$  roughly with constant speed between  $t$  and  $T$ , hardly meeting any Poissonian points in this regime and collecting approximately  $x^2/(2(T - t))$  action. Finding the minimum of

$$St + \frac{x^2}{2(T - t)}, \quad t \in [0, T],$$

we obtain that the optimal  $t$  satisfies

$$\frac{x}{t} = \sqrt{-2S}.$$

This nonrigorous argument shows that we can hope that part 4 of the main theorem holds with  $q = \sqrt{-2S}$ .

To make this argument precise we need to control the deviations of the action on  $[0, t]$  from  $St$  and to control the behavior of minimizers very far from the origin where Poissonian points are sparse.

For any connected set  $I \subset \mathbb{R}$  and any time interval  $[t_0, t_1]$  we can define

$$S_I^{t_0, t_1} = S_I^{t_0, t_1}(\omega) = \inf \left\{ \mathcal{A}_\omega^{t_0, t_1}(\gamma) : \gamma(s) \in I \text{ for all } s \in [t_0, t_1] \right\}.$$

Clearly, this function is superadditive:

$$S_I^{t_0, t_2} \geq S_I^{t_0, t_1} + S_I^{t_1, t_2}, \quad t_0 \leq t_1 \leq t_2.$$

The ergodicity of the flow  $(\theta^t)_{t \in \mathbb{R}}$  and Kingman's subadditive ergodic theorem imply that the following random variable is well defined and a.s.-constant :

$$S_I = \lim_{t_1 \rightarrow \infty} \frac{1}{t_1 - t_0} S_I^{t_0, t_1} = \lim_{t_0 \rightarrow -\infty} \frac{1}{t_1 - t_0} S_I^{t_0, t_1} = \lim_{\substack{t_0 \rightarrow -\infty \\ t_1 \rightarrow \infty}} \frac{1}{t_1 - t_0} S_I^{t_0, t_1}.$$

Clearly,  $S_{B_R}$  is a nonincreasing negative function of  $R > 0$ , and we define  $S = \lim_{R \rightarrow \infty} S_{B_R} < 0$ .

**Lemma 6.1.** *Thus defined constant  $S$  satisfies  $S = S_{\mathbb{R}}$ .*

PROOF: Obviously,  $S_{\mathbb{R}} \leq S_{B_R}$  for any  $R$ . Therefore, we only have to prove that  $S_{\mathbb{R}} \geq S$ . Let us take any  $t > 0$  and any path  $\gamma$  realizing  $S_{\mathbb{R}}^{0, t}$ . Taking any  $R > 0$  and decomposing  $\gamma$  into parts that stay inside  $B_R$  and outside  $B_R$ , we see that

$$S_{\mathbb{R}}^{0, t} = \mathcal{A}_\omega^{0, t}(\gamma) \geq S_{B_R}^{0, t} - N([0, t] \times B_R^c).$$

Dividing by  $t$  and taking  $t \rightarrow \infty$ , we obtain

$$S_{\mathbb{R}} \geq S_{B_R} - m(B_R^c).$$

Taking  $R \rightarrow \infty$  finishes the proof of the lemma.  $\square$

**Lemma 6.2.**

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E} S_{B_R}^{0,t}}{t} = S_{B_R}.$$

PROOF: Since

$$0 \geq \frac{S_{B_R}^{0,t}}{t} \geq -\frac{N([0, t] \times \mathbb{R})}{t},$$

the lemma follows by dominated convergence.  $\square$

**Lemma 6.3.** (1) Let  $R \in (0, \infty]$ . If  $S' < S$ , then there are constants  $c = c(S') > 0$  and  $T_0 = T_0(S') > 0$  such that for  $T > T_0$

$$\mathbb{P} \left\{ \inf_{t \geq T} \frac{S_{B_R}^{0,t}}{t} < S' \right\} < e^{-cT}.$$

(2) If  $S' > S$ , then there is a constant  $R_0 = R_0(S')$  with the following property: for every  $R \in [R_0, \infty]$ , there are  $c = c(S', R) > 0$  and  $T_0 = T_0(S', R) > 0$  such that for  $T > T_0$ ,

$$\mathbb{P} \left\{ \sup_{t \geq T} \frac{S_{B_R}^{0,t}}{t} > S' \right\} < e^{-cT}.$$

PROOF: Recall that  $S_{B_R} \downarrow S < 0$ . Since  $S' < S < S_{B_R}$ , Lemma 6.2 allows to choose  $s$  such that

$$(6.1) \quad \mathbb{E} S_{B_R}^{0,s} > s \frac{S_{B_R} + S'}{2}.$$

Then we notice that

$$\begin{aligned} S_{B_R}^{0,t} &\geq S_{B_R}^{0,s}(\omega) + S_{B_R}^{s,2s}(\omega) + \dots + S_{B_R}^{[t/s]s, ([t/s]+1)s}(\omega) \\ &\geq S_{B_R}^{0,s}(\omega) + S_{B_R}^{0,s}(\theta^s \omega) + \dots + S_{B_R}^{0,s}(\theta^{[t/s]s} \omega). \end{aligned}$$

Let us denote the r.h.s. by  $\Sigma_{[t/s]+1}$ . It is the sum of  $[t/s]+1$  i.i.d. nonpositive random variables with finite exponential moments, and with expectations estimated by (6.1).

Since  $\frac{S' t}{[t/s]+1} \rightarrow S' t$  as  $t \rightarrow \infty$ , and  $S' < (S_{B_R} + S')/2$ , the estimate

$$\mathbb{P}\{S_{B_R}^{0,t} < S' t\} \leq K_1 e^{-c_1 t},$$

for all  $S' < S$ , some  $K_1 = K_1(S')$ ,  $c_1 = c_1(S') > 0$ , and all  $t > 0$ , is a consequence of the classical Cramér large deviation estimate. Since  $S_{B_R}^{0,t}$  is nonincreasing in  $t$ , and

$$\inf_{s \in [t, t+1]} (S_{B_R}^{0,s} - S_{B_R}^{0,t}) \geq -N([t, t+1] \times \mathbb{R}),$$

we can use this maximal inequality in a standard way to interpolate between  $t$  and  $t+1$  and obtain

$$\mathbb{P} \left\{ \inf_{s \in [t, t+1]} \frac{S_{B_R}^{0,s}}{s} < S' \right\} \leq K_2 e^{-c_2 t},$$

for all  $S' < S$ , some  $K_2 = K_2(S')$ ,  $c_2 = c_2(S') > 0$ , and all  $t > 0$ . Now first part of the lemma follows. The proof of the second part of the lemma is essentially the same.  $\square$

Now we turn to ruling out paths ending at a large  $x$  and having slopes deviating significantly from  $q$ . For any  $x > 0$ ,  $\varepsilon \in (0, q)$ , and  $R \in (0, x)$  let us denote

$$Q_+(x, \varepsilon, R) = \{(t, y) : y \in (R, x), t < 0, y < x + t(q + \varepsilon)\},$$

$$Q_-(x, \varepsilon, R) = \{(t, y) : y \in (R, x), t < 0, y > x + t(q - \varepsilon)\}.$$

**Lemma 6.4.** *For each  $\varepsilon > 0$ , there is random variable  $R$  such that*

- (1) *For any  $x > 2R$ , a path  $\gamma$  inside  $Q_+$  connecting a point  $(t, y) \in \partial Q_+(x, \varepsilon, R)$  with  $y > R$  to  $(0, x)$  and  $\dot{\gamma}(0) > q + 2\varepsilon$  cannot be a minimizer.*
- (2) *For any  $x > 2R$ , a path  $\gamma$  inside  $Q_-$  connecting a point  $(t, y) \in \partial Q_-(x, \varepsilon, R)$  with  $y > R$  to  $(0, x)$  and  $\dot{\gamma}(0) < q - 2\varepsilon$  cannot be a minimizer.*

PROOF: First, we notice that condition (4.1) implies that the set  $\{(t, x) : -x < t < 0\}$  contains finitely many Poissonian points with probability 1. Therefore, we can define a random variable  $r_0$  such that with probability 1, there are no Poissonian points in the set

$$A = \{(t, x) : x > r_0, -x < t < 0\}.$$

Let us now define  $\rho = q/(2 + q)$  and, for any  $x$ ,

$$A_x = [-\rho x/q, 0] \times [x - \rho x, x].$$

It is easy to check that  $(-\rho x/q, x - \rho x) \in A$  for sufficiently large values of  $x$ . Therefore, for these values of  $x$ ,  $A_x \subset A$ . We conclude that there is a random number  $r_1$  such that for  $x > r_1$ , there are no Poissonian points in  $A_x$ .

Let us now take a path  $\gamma$  satisfying the conditions of part 1 of the Lemma. We would like to compare this path to the straight line segment connecting  $(t, y)$  and  $(0, x)$ .

**Lemma 6.5.** *Consider points  $(t_0, x_0)$ ,  $(t_1, x_1)$ ,  $(t_2, x_2)$  satisfying  $t_0 < t_1 < t_2$ . The free action (i.e. the action without taking account the contribution from the Poissonian points) of the path connecting these points bounded below by*

$$\frac{(x_2 - x_0)^2}{2(t_2 - t_0)} + \frac{2}{t_2 - t_0}(x_1 - \bar{x})^2,$$

where

$$\bar{x} = \frac{x_2(t_1 - t_0) + x_0(t_2 - t_1)}{t_2 - t_0},$$

so that  $|x_1 - \bar{x}|$  is the distance from  $(t_1, x_1)$  to the straight line connecting  $(t_0, x_0)$  and  $(t_2, x_2)$ , measured along the  $x$ -axis.

PROOF: The free action minimizing path consists of two straight line segments connecting  $(t_0, x_0)$  to  $(t_1, x_1)$  and  $(t_1, x_1)$  to  $(t_2, x_2)$ . The resulting action is a quadratic polynomial in  $x_1$ :

$$\begin{aligned} f(x_1) &= \frac{(x_1 - x_0)^2}{2(t_1 - t_0)} + \frac{(x_2 - x_1)^2}{2(t_2 - t_1)} \\ &= \frac{t_2 - t_0}{2(t_2 - t_1)(t_1 - t_0)} (x_1 - \bar{x})^2 + \frac{(x_2 - x_0)^2}{2(t_2 - t_0)}, \end{aligned}$$

and the estimate

$$\frac{t_2 - t_0}{2(t_2 - t_1)(t_1 - t_0)} = \frac{(t_2 - t_1) + (t_1 - t_0)}{2(t_2 - t_1)(t_1 - t_0)} \geq \frac{2}{(t_2 - t_1) + (t_1 - t_0)} = \frac{2}{t_2 - t_0}$$

completes the proof.  $\square$

Let us denote by  $(s, z)$  the Poissonian point that is connected by the last segment of path  $\gamma$  to  $(0, x)$ . To apply Lemma 6.5, we must estimate the distance from  $(s, z)$  to the straight line connecting  $(t, y)$  and  $(0, x)$  measured along the  $x$ -axis, i.e.,  $x - z + s(q + \varepsilon)$ . Since there are no Poissonian points in  $A_x$ , we have  $(s, z) \in Q_+ \setminus A_x$  and, consequently,  $z \leq x - \rho x$ . Since  $\dot{\gamma}(0) > q + 2\varepsilon$ , we have  $(x - z)/(-s) > q + 2\varepsilon$ . The minimum of  $x - z + s(q + \varepsilon)$  under these restrictions is attained at  $(-\rho x/(q + 2\varepsilon), x - \rho x)$  and equals  $\varepsilon \rho x/(q + 2\varepsilon)$ . We also have  $|t| < (x - R)/(q + \varepsilon)$ . Therefore, Lemma 6.5 implies that the action gain of  $\gamma$  compared to the straight line motion between  $t$  and 0 is at least

$$(6.2) \quad \frac{2(q + \varepsilon)}{x - R} \frac{\varepsilon^2 \rho^2 x^2}{(q + 2\varepsilon)^2} \geq K(\varepsilon)x, \quad x > (2R) \vee r_1(\omega)$$

for some  $K(\varepsilon)$ .

Now we must estimate the effect of Poissonian points. We use Lemma 5.4 to find  $R_0 = R_0(\omega)$  such that  $N([-t, 0] \times B_{R_0}^c) < tK(\varepsilon)q/2$  for all  $t > 0$ . Since the time component of any point in  $Q_+$  is bounded by  $x/q$  in absolute value, we see that if  $R > R_0$ , then there are at most  $(x/q)K(\varepsilon)q/2 < K(\varepsilon)x/2$  Poissonian points in  $Q_+$ . Therefore, the reduction of action due to visits to Poissonian points does not exceed  $K(\varepsilon)x/2$ , and cannot compensate for the action gain computed in (6.2). Therefore, if we choose  $R > R_0 \vee r_1$ , then for any  $x > 2R$  the straight line segment from  $(t, y)$  to  $(0, x)$  is more efficient than any path  $\gamma$  satisfying the imposed requirements, so  $\gamma$  cannot be a minimizer. The proof of the first part of the lemma is complete. The proof of the second part is similar and we omit it.  $\square$

**Lemma 6.6.** *For any  $\varepsilon > 0$  there are random variables  $R > 0$  and  $X > 0$  such that if  $x > X$  and  $\tau > (x - R)/(q - \varepsilon)$ , then no path  $\gamma \in M_{-\tau, R}^{0, x}$  can satisfy*

$$(6.3) \quad \gamma(s) > (x + (q - \varepsilon)s) \vee R, \quad s \in (-\tau, 0].$$

PROOF: We begin with taking  $\delta > 0$  (to be chosen later) and using Lemma 5.4 to find  $R_0$  such that for all  $\tau > (x - R)/(q - \varepsilon)$  and all  $R > R_0$ , the action of

any path  $\gamma$  satisfying (6.3), connecting  $(-\tau, R)$  to  $(0, x)$ , and staying outside of  $B_R$  for all times in  $(-\tau, 0]$ , is at least

$$\frac{(x-R)^2}{2\tau} - \delta\tau.$$

To prove that  $\gamma \notin M_{-\tau, R}^{0, x}$ , let us find a better path  $\tilde{\gamma}$  in  $\Gamma_{-\tau, R}^{0, x}$ . First, we will choose  $\tilde{\gamma}$  so that  $\tilde{\gamma}|_{[-\tau+1, -[(x-R)/q]-2} \in M_{-\tau+1, B_R}^{-[(x-R)/q]-2, B_R}$ . Then we denote  $x_1 = \tilde{\gamma}(-[(x-R)/q]-2)$  and  $x_2 = \tilde{\gamma}(-\tau+1)$ . The remaining parts of  $\tilde{\gamma}$  are straight line segments connecting  $(-\tau, R)$  to  $(-\tau+1, x_2)$ ,  $(-[(x-R)/q]-2, x_1)$  to  $(-[(x-R)/q]-1, R)$ , and  $(-[(x-R)/q]-1, R)$  to  $(0, x)$ .

The action of this path is at most

$$\frac{(x-R)^2}{2([x-R]/q+1)} + \frac{(2R)^2}{2} + S_{B_R}^{0, \tau-1-([(x-R)/q]+2)}(\theta^{-[(x-R)/q]-2}\omega) + \frac{(2R)^2}{2}.$$

We want to exclude the situation where

$$(6.4) \quad \mathcal{A}^{-\tau, 0}(\tilde{\gamma}) \geq \mathcal{A}^{-\tau, 0}(\gamma).$$

Suppose (6.4) holds. Then

$$\frac{(x-R)^2}{2([x-R]/q+1)} + (2R)^2 + S_\tau \geq \frac{(x-R)^2}{2\tau} - \delta\tau,$$

where we denoted  $S_\tau = S_{B_R}^{0, \tau-3-[(x-R)/q]}(\theta^{-[(x-R)/q]-2}\omega)$  for brevity. This can be rewritten as

$$(6.5) \quad \frac{S_\tau}{\tau - 3 - [(x-R)/q]} \geq U,$$

where

$$U = \frac{\frac{(x-R)^2}{2} \left( \frac{1}{\tau} - \frac{1}{[(x-R)/q]+1} \right) - \delta\tau - (2R)^2}{\tau - 3 - [(x-R)/q]}.$$

To apply large deviation estimates from Lemma 6.3, we need to estimate  $U$  and the length of time interval in the definition of  $S_\tau$ . Since  $\gamma$  satisfies (6.3) for  $s \in [-\tau, 0]$ , we have  $\tau > (x-R)/(q-\varepsilon)$ . For sufficiently small  $\varepsilon$ ,

$$(6.6) \quad \tau - 3 - [(x-R)/q] \geq \frac{x-R}{q-\varepsilon} - \frac{x-R}{q} - 4 \geq \frac{(x-R)\varepsilon}{q(q-\varepsilon)} - 4 \geq \frac{(x-R)\varepsilon}{2q^2}.$$

To estimate  $U$ , we first notice that, due to (6.6), there is a random variable  $X_1(R, \varepsilon, \delta)$  such that  $x > X_1$  implies

$$(6.7) \quad \frac{\delta\tau + (2R)^2}{\tau - 3 - [(x-R)/q]} < 2\delta.$$

For the same reason, there is a random variable  $X_2(R, \varepsilon, \delta)$  such that  $x > X_2$  implies

$$\begin{aligned}
 \frac{\frac{(x-R)^2}{2} \left( \frac{1}{\tau} - \frac{1}{[(x-R)/q]+1} \right)}{\tau - 3 - [(x-R)/q]} &= \frac{(x-R)^2}{2\tau([(x-R)/q]+1)} \cdot \left( -\frac{\tau-1-[(x-R)/q]}{\tau-3-[(x-R)/q]} \right) \\
 &\geq -\frac{(x-R)q}{2\tau}(1+\delta). \\
 (6.8) \quad &\geq -\frac{q(q-\varepsilon)}{2}(1+\delta),
 \end{aligned}$$

where the last inequality follows from  $\tau > (x-R)/(q-\varepsilon)$ .

Combining (6.7) and (6.8) and choosing  $\delta$  sufficiently small we see that (with the choices of  $R$  and  $x$  described above)

$$(6.9) \quad U > -\frac{q(q-\varepsilon/2)}{2}.$$

Notice that if  $k$  is sufficiently large, all the above estimates apply uniformly for all  $x \in [R+kq, R+(k+1)q]$  and all  $\tau \geq (x-R)/(q-\varepsilon)$ .

Let us denote by  $B_k$  the event that for some  $x \in [R+kq, R+(k+1)q]$  and some  $\tau \geq (x-R)/(q-\varepsilon)$  there is a path  $\gamma \in M_{-\tau,R}^{0,x}$  satisfying (6.3) for  $s \in [-\tau, 0]$ . The definition of  $q$ , inequality (6.9), and Lemma 6.3 imply that for some  $c > 0$  and all sufficiently large  $k$

$$\mathbb{P}(B_k) < e^{-ck}.$$

Now the Borel–Cantelli Lemma implies that with probability 1 only finitely many events  $B_k$  happen, and the proof is complete.  $\square$

**Lemma 6.7.** *There are positive random variables  $R, X$ , and  $(T_x)_{x>0}$  such that if  $x > X$  and  $\tau > T_x$ , then for any  $y \in \mathbb{R}$ , no path  $\gamma \in M_{-\tau,y}^{0,x}$  can satisfy (6.3).*

PROOF: If for some  $y$  a path  $\gamma \in \Gamma_{-\tau,y}^{0,x}$  satisfies (6.3) and the time  $\tau' = \sup\{s : \gamma(s) \leq R\}$  is well-defined then we can apply Lemma 6.6 with  $\tau$  replaced by  $\tau'$  to see that  $\gamma$  cannot be a minimizer for appropriately chosen of  $R$  and  $X$ .

Let us fix  $\delta \in (0, -S)$ . Due to Lemma 5.4, we can choose  $R$  large enough to ensure that  $\mathcal{A}^{-\tau,0} > -\delta\tau$  for any  $\gamma$  satisfying  $\gamma(s) > R$  for all  $s \in [-\tau, 0]$ . On the other hand, the optimal action is asymptotic to  $S\tau$  as  $\tau \rightarrow \infty$ , so  $\gamma$  cannot be a minimizer for large values of  $\tau$ .  $\square$

**Lemma 6.8.** *There are positive random variables  $R, X$ , and  $(T_x)_{x>0}$  such that for  $x > X$  and  $T > T_x$ , and any  $y \in \mathbb{R}$  no  $\gamma \in M_{-T,y}^{0,x}$  can satisfy*

$$(6.10) \quad \gamma(s) < x + s(q + \varepsilon), \quad s \in [-(x-R)/(q+\varepsilon), 0].$$

PROOF: We need an auxiliary path  $\bar{\gamma} \in \Gamma_{-(1+\varepsilon)[(x-R)/q], -R}^{0,x}$ . This special path consists of three straight line segments connecting consecutively  $(-(1+\varepsilon)[(x-R)/q], -R)$  to  $(-[(x-R)/q], -R)$  to  $(-[(x-R)/q] + 1, R)$  to  $(0, x)$ .

**Lemma 6.9.** *There are positive random variables  $R$  and  $X$  such that for any  $x > X$ , any  $T > -(1 + \varepsilon)[(x - R)/q]$ , and any  $y \in \mathbb{R}$ , any  $\gamma \in M_{-T,y}^{0,x}$  satisfying (6.10) intersects  $\bar{\gamma}$ .*

PROOF: Denote by  $B_k$  the event that there are  $y \in \mathbb{R}$  and  $\gamma \in M_{-T,y}^{-t,-R}$  for some  $t < k$  and  $T > (1 + \varepsilon)k$  such that  $\gamma(s) < -R$  for all  $s \in [-T, -t]$ .

Lemmas 5.4 and 6.3 imply that for sufficiently large  $R$ , for some constants  $c_1, c_2 > 0$  and all  $k$ ,  $P(B_k) \leq c_1 e^{-c_2 \varepsilon k}$ . The Borel–Cantelli Lemma implies that with probability 1, only finitely many events  $B_k$  happen.

Clearly, if there is a path  $\gamma$  satisfying the conditions of the lemma and not intersecting  $\bar{\gamma}$ , then  $B_k$  holds for  $k = [(x - R)/q]$ . Since only finitely many  $B_k$  can hold, a path with these properties is impossible for sufficiently large  $x$ .  $\square$

**Lemma 6.10.** *There are random variables  $R$  and  $X$  such that for  $x > X$  and any path  $\gamma$  satisfying (6.10) and intersecting  $\bar{\gamma}$  at some time  $-\tau$ ,*

$$(6.11) \quad \mathcal{A}^{-\tau,0}(\gamma) > \mathcal{A}^{-\tau,0}(\bar{\gamma}).$$

PROOF: First let us consider the possibility that  $\tau < [(x - R)/q]$ . We denote  $\nu = \inf\{s : \gamma(-s) = R\}$ . For any  $\delta > 0$  there is  $R$  such that for  $x > R$ ,

$$(6.12) \quad \mathcal{A}^{-\tau,0}(\gamma) \geq \frac{(x - R)^2}{2\nu} + S_{\mathbb{R}}^{-[\frac{x-R}{q}]+1,-\nu} - \delta\nu.$$

On the other hand

$$\mathcal{A}^{-\tau,0}(\bar{\gamma}) \leq \frac{(x - R)^2}{2([\frac{x-R}{q}] - 1)} + \frac{(2R)^2}{2}.$$

If (6.11) is violated, the last two inequalities imply:

$$(6.13) \quad \frac{S_{\mathbb{R}}^{-[\frac{x-R}{q}]+1,-\nu}}{[\frac{x-R}{q}] - 1 - \nu} < -\frac{(x - R)^2}{2\nu([\frac{x-R}{q}] - 1)} + \frac{\delta\nu}{[\frac{x-R}{q}] - 1 - \nu} + \frac{2R^2}{[\frac{x-R}{q}] - 1 - \nu}.$$

From (6.10) we know that  $\nu < (x - R)/(q + \varepsilon)$ . We can use this to derive that the second term in the r.h.s. is bounded by  $K\delta/\varepsilon$  for a constant  $K > 0$  and the third term converges to 0 as  $x \rightarrow \infty$ . Choosing  $\delta$  sufficiently small, then choosing  $R$  so that (6.12) holds, we conclude that for sufficiently large  $x$ , the r.h.s. does not exceed  $-q(q + \varepsilon/2)/2$ . Now the large deviation estimate of Lemma 6.3 and the Borel–Cantelli lemma imply that (6.13) can hold true only for a bounded set of  $x$ .

Now we have to exclude the paths  $\gamma$  that cross  $\bar{\gamma}$  for the first time at  $-R$ . By considering a smaller value of  $\delta$  in the above reasoning, it is easy to strengthen it and conclude that that there is  $\Delta > 0$  such that for sufficiently large  $x$ , all paths  $\gamma$  satisfying this restriction satisfy also

$$(6.14) \quad \mathcal{A}^{-[(x-R)/q],0}(\gamma) > \mathcal{A}^{-[(x-R)/q],0}(\bar{\gamma}) + \Delta(x - R).$$

On the other hand, denoting  $I_{\varepsilon,R,x} = \left[ -(1+\varepsilon)\left[\frac{x-R}{q}\right], -\left[\frac{x-R}{q}\right] \right]$

$$\inf_{t \in I_{\varepsilon,R,x}} \mathcal{A}^{t,[x-R]/q}(\gamma) \geq -N(I_{\varepsilon,R,x} \times (-\infty, -R])$$

Suppose  $R$  is chosen so that  $\mathbb{E}N(I_{\varepsilon,R,x} \times (-\infty, -R]) < \Delta(x-R)/2$ . Then probability that for some  $x \in [k, k+1]$  there is a path  $\gamma$  with  $\gamma(0) = x$  satisfying (6.14) and violating (6.11) decays exponentially in  $k$ . An application of the Borel–Cantelli finishes the proof.  $\square$

Part 4 of the main theorem follows now from Lemmas 6.4, 6.6, 6.7, and 6.8.

## 7. THE GLOBAL SOLUTION AS A ONE-POINT ATTRACTOR

In this section we prove part 5 of the main theorem.

Let us denote by  $M_{t_0, \mathbb{R}, V}^{t_1, x}$  the set of minimizers of (2.3).

**Lemma 7.1.** *Suppose  $V$  satisfies (4.2). Then for any  $L > 0$ , there is a random variable  $R_0 > 0$  such that for all  $t > 0$  and all  $x \in B_L$ , any  $\gamma \in M_{0, \mathbb{R}, V}^{t, x}$  satisfies  $\gamma(0) \in B_{R_0}$ .*

PROOF: Property (4.2) implies that there is  $\alpha \in (0, q)$  such that

$$(7.1) \quad V(y) > -\alpha y \text{ for sufficiently large } y > 0.$$

Let us take a small  $\delta > 0$  to be chosen precisely later and use Lemmas 6.1 and 5.4 to find  $h > L$  such that  $S_{B_h} < S + \delta$  and  $N(B_h^c \times [0, s]) < \delta s$  for all  $s > 0$ . Let us consider  $y \in [k, k+1]$  for large values of  $k \in \mathbb{N}$  and estimate the action of a path  $\gamma \in \Gamma_{0,y}^{t,x}$ . Since  $x \in B_L \subset B_h$ , we can define

$$\tau = \inf\{s : \gamma(s) \in B_h\}.$$

The complete action of this path on  $[0, \tau]$  satisfies

$$\mathcal{A}_V^{0,\tau}(\gamma) \geq -\alpha(k+1) - \delta\tau + \frac{(k-h)^2}{2\tau}.$$

On the other hand, there is a number  $C(h)$  such that the optimal path  $\gamma_h \in M_{0, B_h}^{\tau, h}$  satisfies

$$\mathcal{A}_V^{0,\tau}(\gamma_h) \leq C(h) + S_{B_h}^{0,\tau-1}.$$

Therefore, if  $\gamma$  is optimal, then

$$-\alpha(k+1) - \delta\tau + \frac{(k-h)^2}{2\tau} \leq C(h) + S_{B_h}^{0,\tau-1}.$$

According to the definition of  $S_{B_h}$ , there is  $\tau_{h,\delta}$  such that if  $\tau > \tau_{h,\delta}$  then  $S_{B_h}^{0,\tau-1} + C(h) \leq (S_{B_h} + \delta)\tau \leq (S + 2\delta)\tau$ . Therefore, if for optimal  $\gamma$ ,  $\tau > \tau_{h,\delta}$ , then

$$(7.2) \quad (S + 3\delta)\tau - \frac{(k-h)^2}{2\tau} \geq -\alpha(k+1).$$

Elementary calculus shows that the global maximum of the l.h.s. in  $\tau$  is achieved at  $\tau^* = (k-h)/\sqrt{2(-S-3\delta)}$  and equals  $-(k-h)\sqrt{-2S-6\delta}$ .

Since  $\alpha \in (0, \sqrt{-2S})$ , inequality (7.2) will be violated for large values of  $k$  if we choose sufficiently small  $\delta$ .

We conclude that with this choice of  $\delta$  and  $h$ , for sufficiently large  $y$  no path  $\gamma \in M_{0,\mathbb{R},V}^{t,x}$  with  $\gamma(0) = y$  can have  $\tau > \tau_{h,\delta}$ . On the other hand, if  $\tau \leq \tau_{h,\delta}$ , then

$$\mathcal{A}_V^{0,\tau}(\gamma) \geq \frac{(k-h)^2}{2\tau_{h,\delta}} - N([0, \tau_{h,\delta}] \times \mathbb{R}) > V(h)$$

for sufficiently large  $k$ , and such a path cannot be a minimizer since  $V(h)$  is the complete action on  $[0, \tau]$  for the trajectory staying at  $h$ .

The case of  $y \in [-k-1, -k]$  is treated similarly.  $\square$

PROOF OF PART 5 OF THEOREM 4.1: Let us take any two initial conditions  $v_1 = V'_1, v_2 = V'_2$  such that  $V_1$  and  $V_2$  satisfy (4.2). Then there is  $\alpha \in (0, q)$  such that (7.1) holds for  $V = V_1$  and  $V = V_2$ .

Let us take  $R > L$  given by Lemma 5.6 and  $R_0 = R_0(\omega)$  given by Lemma 7.1. Due to Lemma 5.6,  $\mathbb{P}\{r^\pm < R\} > 0$ , where  $r^\pm$  was introduced in Lemma 5.3. That Lemma, along with the ergodicity of the flow  $(\theta^t)$  and Poincaré Recurrence Theorem, allows to find  $n > 0$  such that  $r^\pm(\theta^{-n}\omega) < R$  and  $\tau^\pm = \tau_{R_0}^\pm(\theta^{-n}\omega) < n$ .

If  $V = V_1$  or  $V = V_2$ , then for any  $x \in B_{R_0}$  and for sufficiently large  $t$ , any  $\gamma \in M_{0,\mathbb{R},V}^{t,x}$  must (by Lemma 7.1) belong to  $M_{0,y}^{t,x}$  for some  $y \in B_{R_0}$ , and, consequently, Lemma 5.3 implies  $\gamma(n) \in B_R$ .

Lemma 5.6 and the Poincaré Recurrence Theorem imply that there is  $n' > n$  and a point  $(t^*, x^*)$  such that for sufficiently large  $t$  and for all  $z, x \in B_R$ , every  $\gamma \in M_{n,z}^{t,x}$  passes through  $(t^*, x^*)$ . Therefore, for these values of  $t$  and any  $x \in B_L \subset B_R$ , any two minimizers  $\gamma_1 \in M_{0,\mathbb{R},V_1}^{t,x}$  and  $\gamma_2 \in M_{0,\mathbb{R},V_2}^{t,x}$  pass through  $(t^*, x^*)$ . Therefore

$$M_{0,\mathbb{R},V_1}^{t,x} \big|_{[t^*, t]} = M_{0,\mathbb{R},V_2}^{t,x} \big|_{[t^*, t]},$$

which implies

$$\Phi_\omega^{0,t} v_1 \big|_{B_L} = \Phi_\omega^{0,t} v_2 \big|_{B_L},$$

and the forward attraction follows since one can take  $v_2 = u_\omega$ .

The proof of the backward attraction is similar, and we omit it.

The global solution uniqueness also follows automatically if (4.5) holds with probability 1. If all we know is that (4.5) holds with positive probability, then we can use its invariance under the dynamics and the ergodicity of  $(\theta^t)$  to see that then it holds with probability 1.  $\square$

## 8. GLOBAL MINIMIZERS

A path  $\gamma : \mathbb{R} \rightarrow \mathbb{R}$  is called a global minimizer if  $\gamma|_{[t_0, t_1]} \in M_{t_0, \gamma(t_0)}^{t_1, \gamma(t_1)}$  for any  $t_0, t_1$  satisfying  $t_0 < t_1$ . A global minimizer is called recurrent if there

is  $R > 0$  and a two-sided sequence  $(t_k)_{k \in \mathbb{Z}}$  such that  $\lim_{k \rightarrow \pm\infty} t_k = \pm\infty$  and  $\gamma(t_k) \in B_R$  for all  $k$ .

**Theorem 8.1.** *With probability 1, there is a unique recurrent global minimizer  $\gamma$ .*

**SKETCH OF PROOF:** We can use Lemma 5.6 to derive sequentially that (i) for any  $T_1 > 0$  and for sufficiently large values  $T_2$  the restrictions onto  $[-T_1, T_1]$  of minimizers in  $M_{-T_2, 0}^{T_2, 0}$  can be bounded by the process of localization radii  $r^\pm(\theta^t \omega)$ ,  $t \in [-T_1, T_1]$ ; (ii) for any  $T_1 > 0$ , and for sufficiently large values  $T_2$ , these minimizers pass through common points before  $-T_1$  and after  $T_1$ , and, therefore, coincide between these points. We conclude that as  $T_2 \rightarrow \infty$  the minimizers stabilize on any finite interval, and the restriction of the resulting limiting trajectory  $\gamma = \gamma_\omega$  on any finite time interval is a minimizer. Moreover,  $|\gamma_\omega(t)| < r^\pm(\theta^t \omega)$  and the recurrence property of  $\gamma_\omega$  follows.

If  $\tilde{\gamma}$  is another recurrent global minimizer, then again one can use Lemma 5.6 to prove that there is a sequence of times  $(s_k)_{k \in \mathbb{Z}}$  such that  $\lim_{k \rightarrow \pm\infty} s_k = \pm\infty$  and  $\tilde{\gamma}(s_k) = \gamma_\omega(s_k)$ . Therefore,  $\tilde{\gamma}$  has to coincide with  $\gamma_\omega$ .  $\square$

The following statement can be proven in a similar way:

**Theorem 8.2.** *If  $\tilde{\gamma}$  is one of the one-sided infinite minimizers constructed in Remark 5.2, then there is  $\tau > 0$  such that restrictions of  $\gamma$  and  $\tilde{\gamma}$  on  $(-\infty, \tau]$  coincide.*

This property shows that the global minimizer has superstrong attraction property in the reverse time. In previously considered situations the exponential convergence of one-sided minimizers in the reverse time was a manifestation of hyperbolicity of the global minimizer. In analogy with that case, it is natural to refer to the property of finite-time supercontraction described in Theorem 8.2 as “hyperhyperbolicity”.

**Global minimizers for Burgers equation with spatially periodic forcing.** Let us now change the framework and switch to the Burgers equation with spatially periodic random forcing. One of the questions that has not been answered for the periodic Burgers equation is the fluctuations of the global minimizer. This question was posed to the author by Yakov Sinai. The goal of this section is to prove that for the Poissonian forcing on the circle  $\mathbb{S}^1$ , the unique global minimizer has diffusive behavior.

Let us consider the Burgers dynamics on  $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$  under Poissonian forcing with intensity measure given by  $dt \times m(dx)$  on the cylinder  $\mathbb{R} \times \mathbb{S}^1$  for some Borel measure  $m$  on  $\mathbb{S}^1$ . If we restrict ourselves to the set  $\mathbb{U}_0 = \{u : \int_{\mathbb{S}^1} u = 0\}$ , then the potential  $V$  of any  $v \in \mathbb{U}_0$  is well-defined and we can define the dynamics via (2.2), (2.3) with the only correction that paths are in  $\mathbb{S}^1$ .

The set  $\mathbb{U}_0$  is invariant under this dynamics. In fact, between occurrences of Poissonian points we are solving the usual unforced Burgers equation and

the mean velocity stays constant. On the other hand it is easy to see that the mean velocity is continuous in time even at the time corresponding to the occurrence of a forcing point. A continuous piecewise constant function is constant, and our invariance claim follows.

The theory that was developed above for the Poisson forcing on the line, applies to this case as well, so one has a unique global attracting solution. One also has a unique global minimizer  $\gamma_\omega$  with asymptotic slope 0 corresponding to the mean velocity 0. To formulate the main theorem we must unfold  $\mathbb{S}^1$  onto its universal cover  $\mathbb{R}$  and treat  $\gamma_\omega$  as a continuous path on  $\mathbb{R}$ .

**Theorem 8.3.** *There is a nonrandom number  $D > 0$  such that  $\gamma_\omega(t)/\sqrt{D|t|}$  converges in distribution to the standard Gaussian random variable as  $t \rightarrow \infty$ .*

**PROOF:** The times between occurrences of Poisson points are exponentially distributed. Therefore, the set of all Poisson points  $(t_k(\omega), x_k(\omega))$  such that there are no other Poisson points in  $[t_k - 1, t_k + 1] \times \mathbb{S}^1$  is unbounded in both directions. We agree that  $\dots < t_{-2} < t_{-1} < 0 < t_0 < t_1 < t_2 < \dots$ . It is easy to check that the global minimizer  $\gamma$  passes through all these points on the cylinder (or their lifts on the universal cover).

Let us denote  $\Delta_k t = t_k - t_{k-1}$ ,  $\Delta_k x = x_k - x_{k-1} (\bmod 1)$  and  $\Delta_k \gamma = \gamma(t_k) - \gamma(t_{k-1})$ . Notice that all random variables from sequences  $(x_k)_{k \in \mathbb{Z}}$ ,  $(\Delta_k t)_{k \in \mathbb{Z}}$ , and realizations of Poissonian point field between  $t_{k-1}$  and  $t_k$  are jointly independent. They are also identically distributed within each sequence, the tails of  $\Delta_k t$  are exponential, and the distribution of  $x_k$  is  $m(dx)/m(\mathbb{S}^1)$ .

Since  $\Delta_k \gamma$  is a functional of  $\Delta_k t$ ,  $\Delta_k x$ , and the realization of the Poissonian field between  $t_{k-1}$  and  $t_k$ , the sequence  $(\Delta_k \gamma)_{k \in \mathbb{Z}}$  of identically distributed random variables is 1-dependent (the dependence comes only through  $x_k$  occurring in both  $\Delta_k x$  and  $\Delta_{k+1} x$ )

We know that there is no systematic drift, i.e.,  $(\gamma(t_k) - \gamma(t_0))/(t_k - t_0) \rightarrow 0$  as  $k \rightarrow \infty$ . By the law of large numbers,  $(t_k - t_0)/k \rightarrow h = \mathbb{E}(t_1 - t_0)$ , so  $(\gamma(t_k) - \gamma(t_0))/k \rightarrow 0$ , and  $\mathbb{E}(\gamma(t_1) - \gamma(t_0)) = 0$ .

Therefore, by Bernstein's CLT for  $m$ -dependent random variables, we conclude that the distribution of  $(\gamma(t_k) - \gamma(t_0))/\sqrt{\sigma^2 t}$  converges weakly to the standard Gaussian one, where  $\sigma^2 = \mathbb{E}(\gamma(t_1) - \gamma(t_0))^2$ . Applying the law of large numbers once again we conclude that

$$\frac{\gamma(t)}{\sqrt{\frac{\sigma^2}{h} t}} \xrightarrow{d} \mathcal{N}(0, 1)$$

as  $t \rightarrow \infty$  along the sequence  $(t_k)$ . To finish the proof one has to extend this convergence to all intermediate values of  $t$ , but this is not hard since the tails of  $\Delta_k t$  are exponential. This completes the proof with  $D = \sigma^2/h$ .  $\square$

**Remark 8.1.** It is also possible to prove a functional version of the above CLT with two-sided Wiener measure in the role of the limiting distribution for appropriately normalized global minimizer.

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